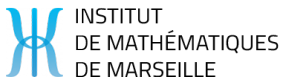


The weak algebraic λ -calculus

JFLA 2025

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January 29, 2025



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Definition (Raw terms of the algebraic λ -calculus)

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Definition (Raw terms of the algebraic λ -calculus)

$$M, N, \dots ::= x \mid \lambda x.M \mid MN \mid aM \mid M + N \mid \mathbf{0}$$

$a \in \mathbb{R}$

Up to some equations: $\underline{M + N} = \underline{N + M}$, $\underline{0M} = \underline{\mathbf{0}}$,
 $\underline{(M + N)P} = \underline{MP + NP}, \dots$

The algebraic λ -calculus lacks operational properties

Definition (Algebraic λ -calculus' reduction)

$$\underline{(\lambda x.s)T} \widetilde{\rightarrow} \underline{s[T/x]} \quad \dots \quad \text{if } \underline{s} \widetilde{\rightarrow} \underline{S'}, a \neq 0, \text{ then } \underline{as + T} \widetilde{\rightarrow} \underline{aS' + T}$$

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Remark

$$\underline{3(\lambda x.x)z} = \underline{(\lambda x.x)z} + 2\underline{(\lambda \mathbf{x}.x)\mathbf{z}} \rightsquigarrow \underline{(\lambda x.x)z} + 2\underline{\mathbf{z}}$$

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Arbitrary fixed-point combinator, arbitrary finite sums:

$$\underline{\infty_M} \stackrel{\text{def}}{=} \underline{(\lambda x.M + xx)(\lambda x.M + xx)} \widetilde{\rightarrow}^* \underline{M + \infty_M \dots}$$

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► Remove $\underline{0M} = \underline{0}$.

In the algebraic λ -calculus

Definition (Pre-normal terms)

Syntactically, $\sum \underline{a_i s_i}$, $a_i \neq 0$ where s_i of the form:

$$\lambda x_1, \dots, x_n. y N_1 \dots N_k$$

where N_i are pre-normal terms.

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How to compute such a term ?

► Freeze the scalars' operations :

$$\mathbb{N}[\{X_a \mid a \in \mathbb{R}\}] : 3 \times 2 + 6 = 12 \rightsquigarrow X_3 X_2 + X_6 \neq X_{12}$$

If the "freezed" M is normalizable, this pre-normal form is unique (Alberti, 2014).

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Theorem (Confluence)

If $\underline{M} \widetilde{\rightarrow}^* \underline{N}_1$ and $\underline{M} \widetilde{\rightarrow}^* \underline{N}_2$, then there is $\underline{P}$ such that $\underline{N}_1 \widetilde{\rightarrow}^* \underline{P}$ and $\underline{N}_2 \widetilde{\rightarrow}^* \underline{P}$.

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Theorem (Conservativity (Kerinec et Vaux Auclair, 2023))

If R has no negative elements, for any M, N pure λ -terms, $M \rightarrow_{\Lambda}^* N$ iff $\underline{M} \widetilde{\rightarrow}^* \underline{N}$.

Weak algebraic reduction

Definition (Weak algebraic λ -calculus' reduction)

$$\underline{(\lambda x.s)T} \xrightarrow{\square} \underline{s[T/x]} \quad \dots \quad \text{if } \underline{s} \xrightarrow{\square} \underline{S'}, \text{ then } \underline{as + T} \xrightarrow{\square} \underline{aS' + T}$$

Weak algebraic reduction

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$$(\lambda x.s)T \xrightarrow{\square} s[T/x] \quad \dots \quad \text{if } \underline{s} \xrightarrow{\square} \underline{S'}, \text{ then } \underline{as} + T \xrightarrow{\square} \underline{aS'} + T$$

Theorem (Confluence)

Let M, N_1, N_2 be weak algebraic λ -terms. If $M \xrightarrow{\square}^* N_1$ and $M \xrightarrow{\square}^* N_2$, then there is P such that $N_1 \xrightarrow{\square}^* P$ and $N_2 \xrightarrow{\square}^* P$.

Theorem (Conservativity)

For any rig R and M, N pure λ -terms, $M \rightarrow_{\Lambda}^* N$ iff $M \xrightarrow{\square}^* N$.

Normalization

Definition (Must-normalization)

A term $\sum \underline{a_i s_i}$ is must-normal if each $\underline{s_i}$ is normal.

Definition (May-normalization)

A term $\sum \underline{a_i s_i}$ is may-normal if there is a i such that $\underline{s_i}$ is normal.

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Error: This expression has type 'a -> 'b but an expression was expected of type 'a
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Figure: A non-typeable normal term

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How about $x : (\alpha \rightarrow \beta) \cap \alpha$?

Type system $\mathcal{D}\Omega_+$

Definition (Types of system $\mathcal{D}\Omega_+$)

$$\alpha, \beta, \dots ::= o \mid \alpha \rightarrow \beta \mid \alpha \cap \beta \mid \Omega$$

$$\begin{array}{c} \frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x. M : \alpha \rightarrow \beta} (\lambda) \quad \frac{\Gamma \vdash M : \alpha \quad \Gamma \vdash N : \alpha}{\Gamma \vdash M + N : \alpha} (+) \\ \frac{\Gamma \vdash M : \alpha}{\Gamma \vdash aM : \alpha} (\text{Scal}) \quad \frac{\Gamma \vdash M : \alpha \quad \Gamma \vdash M : \beta}{\Gamma \vdash M : \alpha \cap \beta} (\cap) \quad \frac{}{\Gamma \vdash M : \Omega} (\Omega) \\ \dots \end{array}$$

Figure: System $\mathcal{D}\Omega_+$ for the weak algebraic λ -calculus (Pagani, Tasson, Vaux, 2016)

Must-normalizability characterization

Theorem

If M is must-normalizable, then M is typable in system $\mathcal{D}\Omega_+$ ($\neq \Omega$).

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Example

$$\begin{aligned} ((\lambda x.x)z)(\lambda y.y)z &\hat{\rightarrow}_{\text{lm}} ((\lambda x.x)z)z \\ (\lambda x.(\lambda y.y)x)z &\hat{\rightarrow}_{\text{lm}} (\lambda x.x)z \end{aligned}$$

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Theorem

M is must-normalizable iff M is strongly normalizing for $\hat{\rightarrow}_{lm}$.

Non-idempotent intersection types

Example

$$\left\{ \begin{array}{l} x:\gamma \rightarrow \alpha \rightarrow \beta \cap \gamma \rightarrow \alpha \\ y:\gamma \end{array} \right\} \vdash (xy)xy : \beta$$
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Definition (Types of system $\mathcal{W}_+$)

$$\alpha, \beta, \dots ::= o \mid \bar{\alpha} \rightarrow \beta \quad \bar{\alpha}, \bar{\beta}, \dots ::= [\alpha_1, \dots, \alpha_n]$$

Where $[\alpha_1, \dots, \alpha_n]$ is a multiset $\rightsquigarrow \alpha_1 \cap \dots \cap \alpha_n$.

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$$[\alpha_1, \dots, \alpha_n] + [\beta_1, \dots, \beta_k] = [\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_k]$$

$$(\Gamma_1 + \Gamma_2)(x) = \Gamma_1(x) + \Gamma_2(x)$$

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$$(\Gamma_1 + \Gamma_2)(x) = \Gamma_1(x) + \Gamma_2(x) \quad [\alpha, \alpha] \neq [\alpha] !$$

May-normalizability characterization

$$\begin{array}{c}
 \frac{}{x : [\alpha] \vdash x : \alpha} \text{ (}\nu\text{)} \quad \frac{\Gamma_1 \vdash M : \bar{\alpha} \rightarrow \beta \quad \Gamma_2 \vdash! N : \bar{\alpha}}{\Gamma_1 + \Gamma_2 \vdash MN : \beta} \text{ (@)} \\
 \frac{\Gamma \vdash M : \alpha}{\Gamma \vdash M + N : \alpha} \text{ (+)} \quad \frac{\Gamma_1 \vdash M : \alpha_1 \quad \dots \quad \Gamma_n \vdash M : \alpha_n}{\Gamma_1 + \dots + \Gamma_n \vdash! M : [\alpha_1, \dots, \alpha_n]} \text{ (!)} \quad \dots
 \end{array}$$

Figure: System $\mathcal{W}_+$ for the weak algebraic λ -calculus

Lemma

If Π is a derivation of $\Gamma \vdash s : \alpha$ and $\underline{s} \xrightarrow{\square} \underline{T}$, then there is Π' a derivation of $\Gamma \vdash T : \alpha$ such that $|\Pi| > |\Pi'|$.

Theorem

A term M is may-normalizable iff M is typable in system $\mathcal{W}_+$.

The resource λ -calculus

Differential : Linear application ($D; M \cdot N$) **in addition to** MN .

Resource : $\langle \sigma \rangle \bar{\rho}$ only.

The resource λ -calculus

Differential : Linear application ($D_i M \cdot N$) in addition to MN .

Resource : $\langle \sigma \rangle \bar{\rho}$ only.

Definition (Resource terms)

$$\Delta \ni \sigma, \rho, \dots ::= x \mid \lambda x. \sigma \mid \langle \sigma \rangle \bar{\rho} \quad !\Delta \ni \bar{\sigma}, \bar{\rho}, \dots ::= [\sigma_1, \dots, \sigma_n]$$

$$\langle \lambda x. \sigma \rangle [\rho_1, \dots, \rho_n] \rightarrow_r \left(\frac{\partial^n \sigma}{\partial x^n} \cdot [\rho_1, \dots, \rho_n] \right) [0/x]$$

$$\langle \sum \sigma_i \rangle \sum \bar{\rho}_j = \sum \sum \langle \sigma_i \rangle \bar{\rho}_j \quad \left[\dots, \sum_{i_j} \rho_{i_j}^j, \dots \right] = \sum_{i_1, \dots, i_n} [\rho_{i_1}^1, \dots, \rho_{i_n}^n]$$

Example

$$\langle \lambda x. \langle x \rangle [x] \rangle [y, z] \rightarrow_r \langle y \rangle [z] + \langle z \rangle [y]$$

$$\langle \lambda x. x \rangle [] \rightarrow_r \mathbf{0} \quad \langle \lambda x. x \rangle [y, z] \rightarrow_r \mathbf{0}$$

Taylor expansion...

Analytically:

$$f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$$

Syntactically, in the λ -calculus:

$$\tau(MN) = \sum_{n=0}^{+\infty} \frac{1}{n!} \langle \tau(M) \rangle \tau(N)^n$$

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In the ordinary and algebraic λ -calculus:

Definition (Resource vectors)

$u \in \mathbb{R}^{\Delta}$: to each resource terms σ , we associate a coefficient u_{σ} .

... in the weak version

Definition (Resource vectors)

$(R \sqcup \{\perp\})^\Delta$: there may be $u_\sigma = \perp \Rightarrow$ not in the support.

Theorem

For any $M, N \in \Lambda_R$:

If $\underline{M} \xrightarrow{\tau} \underline{N}$, then $\tau(M) \Rightarrow_{\mathcal{B}} \tau(N)$;

If M is normalizable, $\tau(NF(M)) = NF(\tau(M))$.

For any $M, N \in \mathbf{\Lambda}$, $\tau(M) \simeq_{\tau} \tau(N)$ iff $M \simeq_R N$.

Example

τ is not injective:

$$\tau(xx) = \tau(xx + 0x(0x))$$

Results

The weak algebraic λ -calculus : a new formalism for sums of λ -terms:

- ▷ Its reduction is confluent;
- ▶ reduction $\leftarrow$ bisimulates $\rightarrow$ reduction of the pure λ -calculus;
- ▶ Two notions of normalizability $\Leftrightarrow$ typeability;
- ▶ For must-normalizability: – almost – a normalization strategy;
- ▷ Reduction --- simulated $\rightarrow$ Taylor expansion, normal forms and Taylor expansion commutes, **not injective**.

Future works

- ▶ Injectivity in the algebraic λ -calculus $\Rightarrow$ conservativity;
- ▶ Normalization scheme for the algebraic λ -calculus from the weak version;
- ▶ $\text{NF}(\tau(M)) \stackrel{?}{=} \tau(\text{BT}(M))$.

Strongly normalizing terms in the weak algebraic λ -calculus

Example

In the algebraic λ -calculus, with $R = \mathbb{Q}_+$, if $\underline{M} \rightsquigarrow \underline{N}$:

$$\underline{M} = \underline{\frac{1}{2}M + \frac{1}{2}M} \rightsquigarrow \underline{\frac{1}{2}M + \frac{1}{2}N} = \underline{\frac{1}{4}M + \frac{1}{4}M + \frac{1}{2}N} \rightsquigarrow \dots$$

Example

In the weak version, worse:

$$\underline{M} = \underline{M + 0M} \xrightarrow{\square} \underline{M \overset{\curvearrowright}{+} 0N}$$

Algebraic equality

$r \subseteq L_R \times L_R$ is *contextual* if it is reflexive and, if $M r N$ and $P r Q$:

$$aM r aN \quad M + P r N + Q \quad 0 r 0 \quad \lambda x.M r \lambda x.N \quad MP r NQ .$$

Definition (Algebraic equality)

$$0 + M \triangleq M$$

$$a0 \triangleq 0$$

$$M + N \triangleq N + M$$

$$\lambda x.0 \triangleq 0$$

$$(M + N) + P \triangleq M + (N + P)$$

$$\lambda x.aM \triangleq a\lambda x.M$$

$$a(M + N) \triangleq aM + aN$$

$$\lambda x.(M + N) \triangleq \lambda x.M + \lambda x.N$$

$$aM + bM \triangleq (a + b)M$$

$$0M \triangleq 0$$

$$a(bM) \triangleq (ab)M$$

$$(aM)N \triangleq a(MN)$$

$$1M \triangleq M$$

$$(M + N)L \triangleq ML + NL .$$

Weak algebraic λ -terms

Definition (Base and canonical terms)

$$B_R \ni s, t, \dots ::= x \mid \lambda x.s \mid sT \quad C_R \ni S, T, \dots ::= \sum_{i=1}^n a_i s_i$$

s_i pairwise distinct, $a_i \in R$.

Definition (Canonization)

If $\text{can}(M) = \sum_{i=1}^n a_i s_i$ and $\text{can}(N) = \sum_{i=n+1}^{n+p} a_i s_i = T$:

$$\begin{aligned} \text{can}(x) &= 1x & \text{can}(\lambda x.M) &= \sum_{i=1} a_i \lambda x.s_i & \text{can}(MN) &= \sum_{i=1}^n a_i (s_i T) \\ \text{can}(M + N) &= \text{cansum} \left(\sum_{i=1}^{n+p} a_i s_i \right) & \text{can}(aM) &= \sum_{i=1}^n (aa_i) s_i \\ \text{can}(0) &= 0 \end{aligned}$$

System $\mathcal{D}\Omega_+$

$$\begin{array}{c}
 \frac{}{\Gamma, x : \alpha \vdash x : \alpha} (v) \quad \frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x. M : \alpha \rightarrow \beta} (\lambda) \quad \frac{\Gamma \vdash M : \alpha \rightarrow \beta \quad \Gamma \vdash N : \alpha}{\Gamma \vdash MN : \beta} (@) \\
 \frac{\Gamma \vdash M : \alpha \quad \Gamma \vdash N : \alpha}{\Gamma \vdash M + N : \alpha} (+) \quad \frac{\Gamma \vdash M : \alpha}{\Gamma \vdash aM : \alpha} (\text{Scal}) \quad \frac{\Gamma \vdash M : \alpha \quad \Gamma \vdash M : \beta}{\Gamma \vdash M : \alpha \cap \beta} (\cap) \\
 \frac{}{\Gamma \vdash M : \Omega} (\Omega) \quad \frac{\Gamma \vdash M : \alpha \quad \alpha \preceq \alpha'}{\Gamma \vdash M : \alpha'} (\text{sub}) \quad \frac{}{\Gamma \vdash \mathbf{0} : \alpha} (z)
 \end{array}$$

(a) Typing rules

$$\frac{}{\alpha \preceq \alpha} \quad \frac{}{\alpha_1 \cap \alpha_2 \preceq \alpha_i} \quad \frac{\alpha' \preceq \alpha \quad \beta \preceq \beta'}{\alpha \rightarrow \beta \preceq \alpha' \rightarrow \beta'} \quad \frac{}{(\alpha \rightarrow \beta) \cap (\alpha \rightarrow \gamma) \preceq \alpha \rightarrow (\beta \cap \gamma)}$$

(b) Subtyping rules